Cosheaves and Discrete Morse Theory

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1. Introduction

In recent years, topological data analysis (TDA) has unveiled insights into the structure and dynamics of complex datasets. At the heart of TDA lies persistent homology, a tool that discerns topological features of data across multiple scales. However, the computational complexity of persistent homology poses significant challenges, especially when dealing with large datasets. The main goal of this essay is to study how discrete Morse theory can be used to speed up the computation of persistent homology of sequences of cosheaves on a finite simplicial complex. In order to do this we will build up the theory of Čech homology from the ground up and provide introductory exposition for all necessary concepts. Some care has been taken in trying to balance the line between focusing on the concrete case of simplicial complexes and the more general setting of a topological space.

Notation: We use $\subset$ to denote non-strict inclusion, $\varinjlim$ to denote colimit of a diagram, and $\varprojlim$ to denote the limit of a diagram.

2. Cosheaves

In this section we formally define simplicial complexes and cosheaves. We define cosheaves in a more general setting than just for simplicial complexes as we believe this offers a more unifying perspective. Some basic familiarity with category theory is assumed.

2.1. Simplicial complexes.

Definition 2.1 (Abstract simplicial complex). *An **abstract simplicial complex** (ASC) is a subset $K \subset \mathcal{P}(N)$ of the power set of some $N \in \mathbf{Set}$ that is closed under taking subsets. We say that K is finite whenever N is a finite set.*

In the rest of this article we shall always assume that N is finite and just refer to K as an ASC.

Now, for any poset, such as K, there is a standard topology we can put on it.

Definition 2.2 (Alexandrov topology, [GS23]). *Let $(\mathcal{P}, \leqslant)$ be a poset. The **Alexandrov topology** on $(\mathcal{P}, \leqslant)$ is defined as follows: $U \subset \mathcal{P}$ is open if and only if for all $u \in U$, $\mathcal{P}_{u\leqslant} := \{p \in \mathcal{P} \mid u \leqslant p\} \subset U$.*

It is quite straightforward to verify that the collection $\mathcal{B} := \{\mathcal{P}_{u\leqslant}\}_{u\in\mathcal{P}}$ forms a basis for the topology on $(\mathcal{P}, \leqslant)$.

For $\mathcal{P} = K$ we will use the notation K_a to refer to K as a topological space with the Alexandrov topology. We also use $\mathbf{st}(\sigma)$ to denote $K_{\sigma\leqslant}$ and refer to it as the **star of the simplex** σ. Lastly, we use the notation $\mathbf{St}_K$ to refer to the basis $\mathcal{B}$ of K_a.

A concept which shall be useful for us later is the nerve of an open cover.

Definition 2.3 (Nerve of open cover). *Let X be a topological space and $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ an open cover of X indexed by an ordered set Λ. The **nerve** of $\mathcal{U}$ is defined to be the ASC*

$$N(\mathcal{U}) := \{I \subset \Lambda \mid U_I \neq \emptyset \text{ and } |I| < \infty\} \tag{1}$$

where $U_I := \bigcap_{i\in I} U_i$.

2.2. Cosheaves. We now turn our attention to studying cosheaves on $(\mathrm{Open}(K_a), \subset)$, where $(\mathrm{Open}(X), \subset)$ is the poset of open subsets on some topological space X.

Definition 2.4 (Copresheaf). *Given categories $\mathcal{C}$ and $\mathcal{D}$, a **copresheaf** on $\mathcal{C}$ is a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. A morphism of copresheaves $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ is a natural transformation $\phi : \mathcal{F} \to \mathcal{G}$. Given two morphism of copresheaves $\phi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$, the composition $\psi \circ \phi : \mathcal{F} \to \mathcal{H}$ is simply the componentwise composition of ψ and ϕ. We denote the category of copresheaves on $\mathcal{C}$ with values in $\mathcal{D}$ as $\mathbf{CoPsh}_{\mathcal{D}}(\mathcal{C})$.*

Definition 2.5 (Properties of morphism of copresheaves). *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of precosheaves. We say that*

(1) ϕ is a monomorphism, if whenever $\psi, \psi' : \mathcal{H} \to \mathcal{F}$ such that $\phi \circ \psi = \phi \circ \psi'$ then $\psi = \psi'$;[1]
(2) ϕ is an epimorphism, if whenever $\eta, \eta' : \mathcal{G} \to \mathcal{H}$ such that $\eta \circ \phi = \eta' \circ \phi$ then $\eta = \eta'$;
(3) ϕ is an isomorphism, if there exist $\varphi : \mathcal{G} \to \mathcal{F}$ such that $\varphi \circ \phi = 1_{\mathcal{F}}$ and $\phi \circ \varphi = 1_{\mathcal{G}}$.

Remark 2.6. *When $\mathcal{D} = \mathbf{Vec}_k$ then ϕ is a monomorphism/epimorphism if and only if each component of ϕ is injective or surjective respectively.*

Moreover, as vertical composition of natural transformations is defined componentwise, we have that if $\phi : \mathcal{F} \to \mathcal{G}$ is an isomorphism, then ϕ_x must be an isomorphism for all $x \in \mathcal{C}$. We thus have that the inverse of ϕ is ϕ^{-1} where $(\phi^{-1})_x = (\phi_x)^{-1}$.

Before giving the definition of a cosheaf we need to say something about different types of open covers.

[1] By $\circ$ we mean vertical composition of natural transformations.

Definition 2.7 ([Cur19]). *Let* $U \subset X$ *be an open set of some topological space* X *and let* $\mathcal{U} = \{U_i\}_{i \in \Lambda} \subset \mathrm{Open}(X)$ *be a collection of open sets indexed over an ordered indexing set* Λ.

(1) We say that $\mathcal{U}$ *is an* ***open cover*** *of* U *if* $\bigcup_{i \in \Lambda} U_i = U$.

(2) We say that $\mathcal{U}$ *is a* ***Čech cover*** *of* U *if* $\mathcal{U}$ *is an open cover of* U *with the additional property that whenever a finite collection* $\{U_i\}_{i \in I} \subset \mathcal{U}$ *has nonempty intersection* U_I *then* $U_I \in \mathcal{U}$.

Example 2.8. *The open cover* $\mathbf{St}_K$ *is a Čech cover. This is because for two* $\sigma, \tau \in K$ *we either have* $\sigma \cap \tau \notin K$ *in which case* $\mathbf{st}(\sigma) \cap \mathbf{st}(\tau) = \emptyset$, *or* $\sigma \cap \tau \in K$ *in which case we have* $\mathbf{st}(\sigma) \cap \mathbf{st}(\tau) = \mathbf{st}(\sigma \cup \tau)$.

Note that for an open cover $\mathcal{U}$ of X we have that the inclusion of the open cover $\iota_{\mathcal{U}} : \mathcal{U} \to \mathrm{Open}(X)$ can be regarded as a functor $\iota_{\mathcal{U}} : (\mathcal{U}, \subset) \to (\mathrm{Open}(X), \subset)$ of posets.

Definition 2.9 (Cosheaf, [Cur19]). *Let* X *be a topological space and let* $\mathcal{D}$ *be a cocomplete category. A* $\mathcal{D}$*-valued copresheaf on* $\mathcal{F} : \mathrm{Open}(X) \to \mathcal{D}$ *is a* ***cosheaf for*** $\mathcal{U}$ *if* $\mathcal{U}$ *is a Čech cover and the canonical map*

$$\mathcal{F}[\mathcal{U}] := \varinjlim \mathcal{F} \circ \iota_{\mathcal{U}} \to \mathcal{F}(\varinjlim \iota_{\mathcal{U}}) = \mathcal{F}(U) \tag{2}$$

is an isomorphism. Moreover, we say that $\mathcal{F}$ *is a* ***cosheaf*** *if for every Čech cover* $\mathcal{U}$, *the functor* $\mathcal{F}$ *is a cosheaf for* $\mathcal{U}$. *We denote the full subcategory of* $\mathcal{D}$*-valued cosheaves on* X *as* $\mathbf{CoSh}_{\mathcal{D}}(X) \subset \mathbf{CoPsh}_{\mathcal{D}}(X)$. *When* $\mathcal{D}$ *is clear from the context, we shall usually drop it and just write* $\mathbf{CoSh}(X)$.

Remark 2.10. *The definition of a cosheaf above requires that* $\mathcal{D}$ *be cocomplete. If* $\mathcal{D}$ *is the category of finite dimensional vector spaces, then this is not the case.*[2] *Hence, when we speak of finite dimensional vector space valued cosheaves we shall think of these vector spaces as lying inside the bigger category of vector spaces.*

Definition 2.11 (Subcosheaf). *Let* $\mathcal{F}, \mathcal{H} \in \mathbf{CoSh}(X)$. *We say that* $\mathcal{H}$ *is a* ***subcosheaf*** *of* $\mathcal{F}$ *if there exists a monomorphism* $\mathcal{H} \to \mathcal{F}$.

It will often be easier for us to specify a cosheaf on the basis of a topological space rather than all open sets. We therefore require the following lemma:

Proposition 2.12 ([GS23],[Cur19]). *Let* X *be a topological space with basis* $\mathcal{B}$ *and consider* $\mathcal{B}$ *as a poset category. A functor* $\hat{\mathcal{F}} : \mathcal{B} \to \mathcal{D}$ *for some cocomplete category* $\mathcal{D}$ *has a unique (up to isomorphism) extension to a cosheaf* $\mathcal{F} : \mathrm{Open}(X) \to \mathcal{D}$. *That is, we have the following Kan extension diagram*

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\hat{\mathcal{F}}"] \arrow[d, hook, "\iota_{\mathcal{B}}"'] & \mathcal{D} \\ \mathrm{Open}(X). \arrow[ur, dashed, "\exists ! \mathcal{F}"'] & \end{tikzcd}$$

Proof. This is a combination of [GS23, Lemma 2.1.13] and [Cur19, Theorem 4.8]. There is some translation between the cited results to be done to get the full statement in our proposition. As this would require a significant detour into category theory we shall not go into the details of this here. □

Corollary 2.13. *Every functor* $\hat{\mathcal{F}} : \mathbf{St}_K \to \mathbf{Vec}_k$ *determines a unique cosheaf* $\mathcal{F} : K_a \to \mathbf{Vec}_k$ *which we call the* ***cosheafification of*** $\hat{\mathcal{F}}$.

[2]As an example, the direct sum of an $\mathbb{N}$-indexed collection of finite dimensional vector spaces is not finite dimensional.

Thus, as we are only interested in working with ASC, we never have to worry about checking the isomorphism condition in the cosheaf definition.

EXAMPLE 2.14 (Constant cosheaf). *Let $\mathcal{B}$ be a basis for X and $V \in \mathbf{Vec}_k$ a vector space over k. Consider the functor $\mathcal{F} : \mathcal{B} \to \mathbf{Vec}_k$ where $\mathcal{F}(U) = V$ and $\mathcal{F}(U \subset U' = 1_V$. The cosheafification of $\mathcal{F}$ defines a cosheaf $\underline{V}_X : \mathrm{Open}(X) \to \mathbf{Vec}_k$ called a **constant cosheaf** on X.*

REMARK 2.15. *When $X = K_a$, a constant cosheaf $\underline{V}_K$ is a cosheaf such that $\underline{V}_K(\mathbf{st}(\sigma)) = V$ for all $\sigma \in K$ and $\underline{V}_K(\mathbf{st}(\sigma) \subset \mathbf{st}(\tau)) = 1_V$ for all $\tau \subset \sigma$.*

A word about notation: we shall drop the subscript K when talking about a constant cosheaf on K as it is understood from the context.

EXAMPLE 2.16 (Pushforward). *Let $f : X \to Y$ be a morphism of topological spaces and suppose $\mathcal{F}$ is a cosheaf on X. The **pushforward** of $\mathcal{F}$ is the cosheaf $f_*\mathcal{F}$ on Y where for $V \subset Y$ an open set we let*

$$f_*\mathcal{F}(V) = \varinjlim_{f(U)\subset V} \mathcal{F}(U). \tag{3}$$

*Since this is a functorial construction we get a functor $f_* : \mathbf{CoSh}(X) \to \mathbf{CoSh}(Y)$ called the **pushforward functor**.*

EXAMPLE 2.17 (Skyscraper cosheaf). *Let $x \in X$ and $V \in \mathbf{Vec}_k$. The **skyscraper cosheaf** on X at x with value in V, denoted x_*^V is defined to be the pushforward of the constant cosheaf $\underline{V}_{\{x\}}$ on $\{x\}$, i.e., for $U \subset X$ open we have*

$$x_*^V(U) = \begin{cases} V, & \text{if } x \in U \\ 0, & \text{else.} \end{cases} \tag{4}$$

Moreover, for $U \subset U'$, we have

$$\begin{cases} 1_V, & \text{if } x \in U \\ 0, & \text{else.} \end{cases} \tag{5}$$

REMARK 2.18. *In the case where $X = K$ and $\sigma \in K$ we get that the skyscraper cosheaf σ_*^V has the following description: for $\tau \in K$ we have*

$$\sigma_*^V(\mathbf{st}(\tau)) = \begin{cases} V, & \text{if } \sigma \supset \tau \\ 0, & \text{else,} \end{cases} \tag{6}$$

and for $\tau' \subset \tau$, we have

$$\sigma_*^V(\mathbf{st}(\tau) \subset \mathbf{st}(\tau')) = \begin{cases} 1_V, & \text{if } \sigma \supset \tau \\ 0, & \text{else.} \end{cases} \tag{7}$$

DEFINITION 2.19 (Costalk of cosheaf). *Let $\mathcal{F} : \mathrm{Open}(X) \to \mathcal{D}$ be a cosheaf and suppose $x \in X$. The **costalk of** $\mathcal{F}$ **at** x is defined to be*

$$\mathcal{F}_x := \varprojlim_{U \ni x} \mathcal{F}(U). \tag{8}$$

EXAMPLE 2.20. *If $X = K_a$ and $\mathcal{F} : \mathrm{Open}(K_a) \to \mathbf{Vec}_k$ is a cosheaf, then*

$$\mathcal{F}_\sigma = \mathcal{F}(\mathbf{st}(\sigma)) \tag{9}$$

for all $\sigma \in K_a$. This follows because $\mathbf{st}(\sigma)$ is the smallest neighborhood of σ in K_a. An interesting consequence of this is that, in lieu of Corollary 2.13, the cosheaf $\mathcal{F}$ is determined by its costalks which is not the case for a general cosheaf.

2.3. Čech homology. Given a Čech open cover $\mathcal{U}$, of some topological space X, and an $\mathbf{Vec}_k$-valued cosheaf $\mathcal{F} : \mathrm{Open}(X) \to \mathbf{Vec}_k$ for some field k, on $\mathcal{U}$. Then there is a combinatorial homology theory[3] on X called Čech homology.

Definition 2.21 (Čech complex). *Let X be a topological space and $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ a Čech open cover of X with some ordered indexing set Λ. Suppose $\mathcal{F} : \mathrm{Open}(X) \to \mathbf{Vec}_k$ is a $\mathbf{Vec}_k$-valued cosheaf on $\mathcal{U}$ for some field k. The* ***Čech complex of*** *X is the chain complex $(\check{C}_\bullet(\mathcal{U};\mathcal{F}), \partial_\bullet)$ where*

$$\check{C}_n(\mathcal{U};\mathcal{F}) := \bigoplus_{|I|=n+1} \mathcal{F}(U_I), \qquad \text{for } I \in N(\mathcal{U}), \tag{10}$$

and $\partial_n : \check{C}_n(\mathcal{U};\mathcal{F}) \to \check{C}_{n-1}(\mathcal{U};\mathcal{F})$ is given on basis elements $s_I \in \mathcal{F}(U_I)$ by

$$\partial_n(s_I) := \sum_{k=0}^{n} (-1)^k r_{U_I^{(k)},U_I}(s_I) \tag{11}$$

where $U_I := U_{i_0} \cap \cdots \cap \widehat{U}_{i_k} \cap \cdots \cap U_{i_n}$, with the hat denoting an absent element, and $r_{U_I^{(k)},U_I}$ is the image of the inclusion map $U_I \subset U_I^{(k)}$ under $\mathcal{F}$.

For Čech homology to make sense we need to actually check that $(\check{C}_\bullet(\mathcal{U};\mathcal{F}), \partial_\bullet)$ is a chain complex, i.e., that $\partial_\bullet$ squares to zero.

Proposition 2.22. *For all $n > 0$, $\partial_{n-1} \circ \partial_n = 0$.*

Proof, [Cur14]. From the functoriality of $\mathcal{F}$ we have a commutative diagram

$$\begin{tikzcd}
& \mathcal{F}\left(U_I^{(j,k)}\right) & \\
\mathcal{F}\left(U_I^{(j)}\right) \arrow[ur, "r_{U_I^{(j,k)},U_I^{(j)}}"] & & \mathcal{F}\left(U_I^{(k)}\right) \arrow[ul, "r_{U_I^{(j,k)},U_I^{(k)}}"'] \\
& \mathcal{F}(U_I). \arrow[ul, "r_{U_I^{(j)},U_I}"] \arrow[ur, "r_{U_I^{(k)},U_I}"'] \arrow[uu, "r_{U_I^{(j,k)},U_I}"] &
\end{tikzcd}$$

Now, suppose $s_I \in \mathcal{F}(U_I)$. Applying ∂ once gives $(-1)^j r_{U_I^{(j)},U_I}(s_I)$ and $(-1)^k r_{U_I^{(k)},U_I}(s_I)$ as two of the components in $\partial(s_I)$. Now, assume without loss of generality that $j < k$, then we must delete the $k-1$st entry of $I - \{j\}$ to get the image of $(-1)^j r_{U_I^{(j)},U_I}(s_I)$ in $\mathcal{F}\left(U_I^{(j,k)}\right)$ when applying ∂ one more time. Thus, the image of $\partial^2(s_I)$ in $\mathcal{F}\left(U_I^{(j,k)}\right)$ is, taking into account the commutativity of the above diagram, given by

$$(-1)^{k-1}(-1)^j r_{U_I,U_I^{(j,k)}}(s_I) + (-1)^k(-1)^j r_{U_I,U_I^{(j,k)}}(s_I) = 0.$$

□

[3]We do not mean homology theory here in the sense of the Eilenberg-Steenrod axioms, but rather in a more informal way. Indeed, Čech homology fails to be a homology theory in this sense and one needs a stronger version of it to get a proper homology theory.

Definition 2.23 (Čech homology). *For a Čech complex $(\check{C}_\bullet(\mathcal{U};\mathcal{F}), \partial_\bullet)$ the* ***Čech homology*** *of X on $\mathcal{U}$ with coefficients in $\mathcal{F}$ is defined to be the homology of the Čech complex, i.e.,*

$$\check{H}_n(\mathcal{U};\mathcal{F}) := H_n(\check{C}_\bullet(\mathcal{U};\mathcal{F})). \tag{12}$$

Proposition 2.24 (Induced map on homology). *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of cosheaves on X. We then have a map on Čech homology $\phi_\bullet : \check{H}_\bullet(\mathcal{U};\mathcal{F}) \to \check{H}_\bullet(\mathcal{U};\mathcal{G})$.*

Proof. To show this we need to construct a chain map $\phi_* : \check{C}_\bullet(\mathcal{U};\mathcal{F}) \to \check{C}_\bullet(\mathcal{U};\mathcal{G})$. Thus, fix some $n \geqslant 0$ and let

$$\phi_n : \bigoplus_{|I|=n+1} \mathcal{F}(U_I) \to \bigoplus_{|I|=n+1} \mathcal{G}(U_I) \tag{13}$$

be the map which on basis elements $s_I \in \mathcal{F}(U_I)$ is defined to be

$$\phi_n(s_I) := \phi_{U_I}(s_I). \tag{14}$$

To see that this is a chain map, we must verify that $\phi_n \circ \partial_{n+1} = \partial_{n+1} \circ \phi_{n+1}$. Thus, let $s_J \in \mathcal{F}(U_J)$. We then have that

$$(\phi_n \circ \partial_{n+1})(s_J) = \phi_n \left(\sum_{k=0}^{n+1} (-1)^k r_{U_J^{(k)}, U_J}(s_J) \right) = \sum_{k=0}^{n+1} (-1)^k \phi_{U_J^{(k)}} \left(r_{U_J^{(k)}, U_J}(s_J) \right) \tag{15}$$

and

$$(\partial_n \circ \phi_{n+1})(s_J) = \partial_n \left(\phi_{U_J}(s_J) \right) = \sum_{k=0}^{n+1} (-1)^k r_{U_J^{(k)}, U_J}(\phi_{U_J}(s_J)). \tag{16}$$

As ϕ is a natural transformation we have that $\phi_{U_J^{(k)}} \circ r_{U_J^{(k)}, U_J} = r_{U_J^{(k)}, U_J} \circ \phi_{U_J}$ and hence the two sums are equal showing that ϕ_* is a chain map. This means we have an induced map $\phi_\bullet$ on Čech homology as desired. □

We now turn our attention to study some examples of abstract simplicial complexes and their Čech homology.

Example 2.25 (Task 3 (1)). *Consider the standard triangulation of the line segment $[0, 1]$ given by $L := 2^{\{0,1\}}$, the power set of $\{0, 1\}$. As Čech homology with coefficients in a constant cosheaf on an ASC is the same thing as simplicial homology, we have that*

$$\check{H}_n(\mathbf{St}_L; \underline{k}) = \begin{cases} k, & \textit{if } n = 0 \\ 0, & \textit{else} \end{cases} \tag{17}$$

which is the same as the homology of a single point. On the other hand, consider the functor $\mathcal{F} : \mathbf{St}_L \to \mathbf{Vec}_k$ defined by

$$\mathcal{F}(\mathbf{st}(\sigma)) = \begin{cases} k, & \textit{if } \sigma \in \{\{0\}, \{0, 1\}\} \\ 0, & \textit{else} \end{cases}$$

and $\mathcal{F}(\mathbf{st}(\{0, 1\}) \subset \mathbf{st}(\{0\})) = 1_k$ with the other inclusion being zero. Using the lexicographical ordering on L we then have the Čech complex

$$0 \longrightarrow k \xrightarrow{\partial_1} k \longrightarrow 0$$

with $\partial_1 = 1_k$. Hence the Čech homology of L with coefficients in the cosheafification of $\mathcal{F}$ is zero which is not the same as the homology of a single point.

EXAMPLE 2.26 (Task 3 (2)). *Consider the standard triangulation of the circle* $S^1 := 2^{\{0,1,2\}} \setminus \{0,1,2\}$. *Again, since Čech homology with coefficients in the constant cosheaf* $\underline{k}$ *is the same thing as simplicial homology, we have that*

$$\check{H}_n(\mathbf{St}_{S^1}; \underline{k}) = \begin{cases} k, & \text{if } n = 0, 1 \\ 0, & \text{else.} \end{cases} \tag{18}$$

On the other hand, let $\sigma = \{0\}$ *and consider the skyscraper cosheaf* σ_*^k. *This has an associated Čech complex given by*

$$0 \to k \to 0 \tag{19}$$

which has the homology of a point

$$\check{H}_n(\mathbf{St}_{S^1}; \sigma_*^k) = \begin{cases} k, & \text{if } n = 0 \\ 0, & \text{else.} \end{cases} \tag{20}$$

EXAMPLE 2.27 (Task 3 (3)). *Consider the standard triangulation of the line segment* $L = 2^{\{0,1\}}$. *Define a cosheaf* $\mathcal{F}$ *on* L *by the following data*

$$\mathcal{F}(\mathbf{st}(\sigma)) = \begin{cases} k^2, & \text{if } \sigma \in \{0, 01\} \\ k, & \text{else.} \end{cases}$$

$$\mathcal{F}(\mathbf{st}(\sigma) \subset \mathbf{st}(\tau)) = \begin{cases} 1_{k^2}, & \text{if } \sigma = \{0,1\} \text{ and } \tau = \{0\} \\ \begin{bmatrix} 1 & 0 \end{bmatrix}, & \text{else.} \end{cases}$$

We then define a strict monomorphism $\phi : \underline{k}_L \to \mathcal{F}$ *by the following data*

$$\phi_\sigma = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \text{if } \sigma \in \{0, 01\} \\ 1_k, & \text{else.} \end{cases} \tag{21}$$

We can picture the situation as in the following diagram

$$\begin{tikzcd} k \arrow[d, "\mathbf{1}"'] & k \arrow[l, "1_k"'] \arrow[r, "1_k"] \arrow[d, "\mathbf{1}"'] & k \arrow[d, "1_k"] \\ k^2 & k^2 \arrow[l, "1_{k^2}"] \arrow[r, "\mathbf{1}^T"'] & k \end{tikzcd}$$

where $\mathbf{1}$ *is the matrix* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. *The Čech complex for* $\mathcal{F}$ *then becomes*

$$0 \to k^2 \xrightarrow{A} k^3 \to 0 \tag{22}$$

where A *is the matrix*

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We see that $\ker A = 0$ *and hence* $\check{H}_\bullet(\mathbf{St}_L; \mathcal{F})$ *has the same homology as* $\check{H}_\bullet(\mathbf{St}_L; \underline{k}_L)$. *Moreover, the induced map on homology is non-zero and hence must be an isomorphism. Since* $\mathbf{1}$ *is not an isomorphism we have that* ϕ *is a strict monomorphism. Thus, we see that there exists a cosheaf* $\mathcal{F}$ *and a strict isomorphism* $\underline{k}_L \to \mathcal{F}$ *which induces isomorphism on Čech homology.*

There is no dual situation to the above example.

Proposition 2.28. *Let $\mathcal{U}'$ be a fixed finite open cover on X such that each $U \in \mathcal{U}'$ is contained in a connected component of X. Then there does not exist a strict epimorphism $\phi : \underline{k}_X \to \mathcal{F}$ for the Čech open cover $\mathcal{U}$ generated by $\mathcal{U}'$ which induces an isomorphism on Čech homology.*

Proof. Let $\phi : \underline{k}_X \to \mathcal{F}$ be a strict epimorphism and let $X = \coprod_i X_i$ be the connected components of X. Note that Čech homology preserves coproducts and so

$$\check{H}_\bullet(\mathcal{U}, \mathcal{F}) = \bigoplus_i \check{H}_\bullet(\mathcal{U}\mid_{X_i}; \mathcal{F}\mid_{X_i}). \tag{23}$$

In particular, the morphism $\phi_\bullet : \check{H}_\bullet(\mathcal{U}, \underline{k}_X) \to \check{H}_\bullet(\mathcal{U}, \mathcal{F})$ splits as the morphism $\oplus_i \phi_i$ where ϕ_i is the morphism $\check{H}_\bullet(\mathcal{U}\mid_{X_i}, \underline{k}_{X_i}) \to \check{H}_\bullet(\mathcal{U}\mid_{X_i}, \mathcal{F}\mid_{X_i})$. We can therefore, without loss of generality, assume that X is connected.

Now, as ϕ is a strict epimorphism, there must exist some $U \in \mathcal{U}'$ such that $\mathcal{F}(U) = 0$. For any other $U' \in \mathcal{U}'$ we then have that there is a sequence of nonempty intersections $(U \cap U_0, U_0 \cap U_1, \ldots, U_n \cap U')$ with $U_i \in \mathcal{U}'$ for $i = 0, \ldots, n$. Defining coefficients appropriately there is then an element $s \in \check{C}_2(\mathcal{U}; \mathcal{F})$ such that $\partial_2(s) = t$ where t is the element in $\check{C}_1(\mathcal{U}; \mathcal{F})$ corresponding to U'. Hence $\check{H}_0(\mathcal{U}; \mathcal{F}) = 0$ and so $\phi_\bullet$ cannot be an isomorphism as $H_0(\mathcal{U}; \underline{k}_X) = k$. □

Corollary 2.29. *There is no strict epimorphism $\underline{k}_K \to \mathcal{F}$ which induces an isomorphism on Čech homology.*

3. Discrete Morse theory

It is often the case that a mathematical object comes with some sort of "filtration". The central idea of persistent homology is to use this filtration to study the homology with respect to it. In our case, this filtration will be in the form of a sequence of cosheaves on a simplicial complex, which we will show induces a sequence of coparameterizations.

3.1. Filtered cosheaves.

Definition 3.1 (Filtration, [nLa24]). *Let $\mathcal{C}$ be a category. A **filtered object** is an object $X \in \mathcal{C}$ equipped with a filtration:*

*(1) A **descending filtration** or **decreasing filtrations** of X is a sequence of morphisms of the form*

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots \to X. \tag{24}$$

Equivalently, it is a functor $F : (\mathbb{N}, \leqslant)^{op} \to \mathcal{C}$ such that $F(0) = X$.

*(2) An **ascending filtration** or **increasing filtration** of X is a sequence of the form*

$$X \to \cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots . \tag{25}$$

Equivalently, it is a functor $F : (\mathbb{N}, \leqslant) \to \mathcal{C}$ such that $F(0) = X$.

When $\mathcal{C}$ is the category of cosheaves K we shall only be interested in decreasing filtrations and additionally require that each morphism in the sequence be a monomorphism, i.e., that each morphism is an inclusion of subcosheaves.

Definition 3.2 (Filtration of cosheaf). *Suppose $\mathcal{F} \in \mathbf{CoSh}(K)$, then a **filtration** of $\mathcal{F}$ is a descending filtration*

$$\cdots \xrightarrow{\phi_{n+2}} \mathcal{F}_{n+1} \xrightarrow{\phi_{n+1}} \mathcal{F}_n \xrightarrow{\phi_n} \mathcal{F}_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} \mathcal{F} \tag{26}$$

in which ϕ_i is a monomorphism for all $i \in \mathbb{N}$. We succinctly write the filtration as $(\mathcal{F}_\bullet, \phi_\bullet)$.

For a filtration $(\mathcal{F}_\bullet, \phi_\bullet)$ we say that it is **exact** if $\operatorname{im}\phi_{n+1} = \ker\phi_n$ for all $n \geqslant 1$ and a **chain complex** if $\operatorname{im}\phi_{n+1} \subset \ker\phi_n$ for all $n \geqslant 1$.

Given a filtration $(\mathcal{F}_\bullet, \phi_\bullet)$ of some some cosheaf $\mathcal{F}$ and a Čech open cover on X we get, from Proposition 2.24, a filtration of vector spaces

$$\cdots \xrightarrow{\check{H}_\bullet\phi_{n+1}} \check{H}_\bullet(\mathcal{U}; \mathcal{F}_n) \xrightarrow{\check{H}_\bullet\phi_n} \check{H}_\bullet(\mathcal{U}; \mathcal{F}_{n-1}) \xrightarrow{\check{H}_\bullet\phi_{n-1}} \cdots \xrightarrow{\check{H}_\bullet\phi_1} \check{H}_\bullet(\mathcal{U}; \mathcal{F}_0). \tag{27}$$

Definition 3.3 (Persistent homology). *Let X be a topological space $(\mathcal{F}_\bullet, \phi_\bullet)$ a filtration of $\mathbf{Vec}_k$-valued cosheaves on X. The **persistent homology** of some Čech open cover $\mathcal{U}$ of X with coefficients in the filtration $(\mathcal{F}_\bullet, \phi_\bullet)$ is defined for $n \geqslant 0$ and $0 \leqslant k \leqslant l$ as*

$$H_n^{k,l}(X; \mathcal{F}_\bullet) := \operatorname{im}(\check{H}_n\phi_k \circ \cdots \circ \check{H}_n\phi_{l-1} \circ \check{H}_n\phi_l). \tag{28}$$

3.2. Partial matchings. Before starting our discussion of partial matchings and compatibility we make some definitions.

For a poset $(\mathcal{P}, \leqslant)$ and $x, y \in \mathcal{P}$ we say that y covers x if and only if $\{z \in \mathcal{P} \mid x < z < y\} = \emptyset$ and denote this by $x \prec y$.

Definition 3.4 (Graded poset). *Let $(\mathcal{P}, \leqslant)$ be a poset. We say that $\mathcal{P}$ is **graded** if it admits a partition $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ into subsets indexed by a dimension so that if $x \prec y$ then $\dim y = \dim x + 1$.*

A simplicial complex K is a graded by the dimension on its simplices and hence we have that $\sigma \prec \tau$ if and only if σ is a codimension one face of τ.

Definition 3.5 (Coparameterization, [CGN15]). *Let $(C_\bullet, \partial_\bullet)$ be a chain complex of k-vector spaces. A **coparameterization** of $(C_\bullet, \partial_\bullet)$ over a graded poset $(\mathcal{P}, \leqslant)$ is a functor $F : (\mathcal{P}, \leqslant)^{op} \to \mathbf{Vec}_k$, where we use the notation $F_{y,x} := F(y \geqslant x)$, such that*

(1) $C_n = \bigoplus_{x\in\mathcal{P}_n} F(x)$,
(2) the part of $\partial_n : C_n \to C_{n-1}$ from $F(y)$ to $F(x)$ is precisely $F_{y,x}$,
(3) $F_{y,x} = 0$ *for* $x \not\prec y$.

*A **morphism of coparameterization** $\theta : F \to G$ is then a natural transformation between the functors F and G.*

Remark 3.6. *When considering $(K, \subset)$ as a poset, we have a contravariant functor $(K, \subset) \to (K_a, \subset)$ which maps $\sigma \in K$ to $\mathbf{st}(\sigma) \in K_a$. Up to signs we therefore have that a cosheaf $\mathcal{F} : K_a \to \mathbf{Vec}_k$ induces a canonical coparameterization F of $\check{C}_\bullet(K_a; \mathcal{F})$ over $(K, \subset)$. More explicitly*

$$F(\sigma) := \mathcal{F}(\mathbf{st}(\sigma)) \tag{29}$$

and

$$F_{\sigma,\tau} = \begin{cases} \mathcal{F}(\mathbf{st}(\sigma) \subset \mathbf{st}(\tau)), & \textit{if } \dim\sigma = \dim\tau + 1 \\ 0, \textit{ else.} \end{cases} \tag{30}$$

Lemma 3.7. *Let $\theta : F \to G$ be a map of coparameterizations of $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial_\bullet)$ respectively. Then we have an induced chain map $\Theta : (C_\bullet, \partial_\bullet) \to (C'_\bullet, \partial_\bullet)$ such that $\Theta_n = \oplus_{x\in\mathcal{P}_n}\theta_x$.*

Proof. Let $\Theta_\bullet$ be given as described. We must then verify that $\Theta_{n-1} \circ \partial_n = \partial'_n \circ \Theta_n$ for all $n \geqslant 1$. For $s \in F(y)$ we then have that the $G(x)$ part is given by

$$\Theta_{n-1}(\partial_n(s)) = \theta_x(F_{y,x}(s)) = G_{y,x}(\theta_y(s)) = \partial'_n(\Theta_n(s)) \tag{31}$$

which shows commutativity as desired. $\square$

Remark 3.8. *For a filtration of cosheaves* $(\mathcal{F}_\bullet, \theta_\bullet)$ *we then have a corresponding filtration of coparameterizations* $(F_\bullet, \Theta_\bullet)$ *which induces the same maps on homology. In particular, we have that the persistent homology can be calculated in terms of the coparameterizations:*

$$H_\bullet^{k,l}(K; \mathcal{F}_\bullet) = \mathrm{im}(H_\bullet\Theta_k \circ \cdots \circ H_\bullet\Theta_{l-1} \circ H_\bullet\Theta_l). \tag{32}$$

Definition 3.9 (Partial matching, [CGN15]). *Let* $(\mathcal{P}, \leqslant)$ *be a graded poset. A* ***partial matching*** *on* $(\mathcal{P}, \leqslant)$ *is a subset* $\Sigma \subset \mathcal{P} \times \mathcal{P}$ *subject to the following axioms:*

(1) ***dimension****: if* $(x, y) \in \Sigma$ *then* $x \prec y$*, and*
(2) ***partition****: if* $(x, y) \in \Sigma$*, then neither* x *nor* y *belongs to any other pair in* Σ*.*

Moreover, Σ *is called* ***acyclic*** *if the transitive closure of the relation* $\lhd$ *defined on pairs in* Σ *by*

$$(x, y) \lhd (x', y') \text{ iff } x' \prec y \tag{33}$$

generates a partial order. An element $x \in \mathcal{P}$ *is called* Σ*-**critical*** *if no pair in* Σ *contains* x*, and we shall denote the set of all critical elements in* $\mathcal{P}$ *by* M*.*

Definition 3.10 (Gradient part, [CGN15]). *Let* Σ *be an acyclic partial matching on* $(\mathcal{P}, \leqslant)$*. A* ***gradient path***[4] γ *of* Σ *is a strictly* $\lhd$*-increasing sequence* $(x_i, y_i)_{i=0}^n \subset \Sigma$ *arranged in the following way*

$$\gamma = x_0 \prec y_0 \succ x_1 \prec y_1 \succ \cdots \succ x_n \prec y_n. \tag{34}$$

For each gradient path γ*, we write* $s_\gamma = x_0$ *and* $t_\gamma = y_n$ *to indicate the source and target of the path respectively.*

Definition 3.11 (Compatible matchings, [CGN15]). *Let* Σ *be an acyclic partial matching on a poset* $(\mathcal{P}, \leqslant)$ *and* F *a coparameterization of a chain complex* $(C_\bullet, \partial_\bullet)$ *over* $(\mathcal{P}, \leqslant)$*. We say that* Σ *is* ***compatible*** *with* F *if for each pair* $(x, y) \in \Sigma$*, the associated linear map* $F_{y,x} : F(y) \to F(x)$ *is invertible.*

Given a cosheaf on $\mathcal{F} : K_a \to \mathbf{Vec}_k$ and an acyclic partial matching Σ on $(K, \subset)$ we have that the induced coparameterization F is compatible with Σ if and only if for all $(\sigma, \tau) \in \Sigma$ we have that $\mathcal{F}(\mathbf{st}(\sigma) \subset \mathbf{st}(\tau))$ is invertible, in which case we say that $\mathcal{F}$ is **compatible** with the partial matching Σ.

Example 3.12. *Consider* S^1 *from Example 2.26 with the constant cosheaf* $\underline{k}_{S^1}$*. Letting* $\Sigma = \{(0, 01), (1, 12)\}$ *we have that* Σ *is an acyclic partial matching. Moreover, as all maps involved are identities, and hence invertible, we see that* $\underline{k}_{S^1}$ *is* Σ*-compatible.*

Now, given a filtration of cosheaves we can then define what it means to be compatible with it.

Definition 3.13. *Let* $(\mathcal{F}_\bullet, \phi_\bullet)$ *be a filtration of* $\mathbf{Vec}_k$*-valued cosheaves on* K*. We say that* $(\mathcal{F}_\bullet, \phi_\bullet)$ *is compatible with an acyclic partial matching* Σ *on* K *if* $\mathcal{F}_n$ *is compatible with* Σ *for all* n*.*

Example 3.14. *Expanding on the previous example we can let* $\mathcal{F}_n = \underline{k^{4-n}}_{S^1}$ *for* $n < 4$ *and* $\underline{0}_{S^1}$ *else. Moreover, we let* $\phi_n : \mathcal{F}_n \to \mathcal{F}_{n-1}$ *be the inclusion into the* n *first factors. We then have that each* $\mathcal{F}_n$ *is* Σ*-compatible and hence the filtration is compatible with* Σ*.*

[4]This is not the standard definition in the literature as what we are defining here technically is a cogradient path.

3.3. Morse data. Suppose we have some acyclic partial matching Σ on $(\mathcal{P}, \leqslant)$ and a coparameterization, F, of some chain complex $(C_\bullet, \partial_\bullet)$ over $(\mathcal{P}, \leqslant)$. Our goal is then to define a new chain complex $(C^\Sigma_\bullet, \partial^\Sigma_\bullet)$ and show that it gives the same homology as the old chain complex.

DEFINITION 3.15 (Multiplicity, [CGN15]). *The **multiplicity** of a gradient path* $\gamma = x_0 \prec y_0 \succ x_1 \prec y_1 \succ \cdots \succ x_n \prec y_n$ *in* Σ *is defined to be*

$$F^\gamma := -F^{-1}_{y_n, x_n} \circ F_{y_{n-1}, x_n} \circ \cdots \circ F_{y_0, x_1} \circ -F^{-1}_{y_0, x_0} \tag{35}$$

which is a linear map $F^\gamma : F(x_0) \to F(y_n)$. *We also call this the **index** of* γ.

In the case where there exists a gradient path γ such that for $x, y \in M$ we have $y \succ s_\gamma$ and $x \prec t_\gamma$, we say that γ **flows** from y to x. We write $y >'_\Sigma x$ whenever there exists at least one such gradient path. As Σ is acyclic, the transitive closure $>_\Sigma$ of $>'_\Sigma$ defines a graded partial order on M.

We can then define the Morse chain complex $(C^\Sigma_\bullet, \partial^\Sigma_\bullet)$.

DEFINITION 3.16 (Morse data, [CGN15]). *The **Morse data** associated to* Σ *consists of the poset* $(M, \leqslant_\Sigma)$ *of critical elements along with a sequence of* k*-vector spaces* $(C^\Sigma_\bullet, \partial^\Sigma_\bullet)$ *where*

$$C^\Sigma_n := \bigoplus_{x \in M_n} F(x) \tag{36}$$

and the part of $\partial^\Sigma_n : C^\Sigma_n \to C^\Sigma_{n-1}$ *corresponding to the map from* $F(y)$ *to* $F(x)$ *is given by*

$$F^\Sigma_{y,x} = F_{y,x} + \sum_\gamma F_{t_\gamma, x} \circ F^\gamma \circ F_{y, s_\gamma} \tag{37}$$

where the sum is taken over all gradient paths γ *flowing from* y *to* x.

The main goal of this section is to prove the following theorem from which all the results about simplicial complexes will follow. We shall closely follow the strategy used in [CGN15], with the change that we will dualize everything.

THEOREM 3.17 (Sköldberg). *Let* F *be a coparameterization of a chain complex* $(C_\bullet, \partial_\bullet)$ *of* k*-vector spaces over a graded poset* $(\mathcal{P}, \leqslant)$ *and let* Σ *be a compatible acyclic matching. Then, the Morse data* $(C^\Sigma_\bullet, \partial^\Sigma_\bullet)$ *is a chain complex coparameterization over* $(M, \geqslant_\Sigma)$ *by* F^Σ. *Moreover, there is an isomorphism of graded vector spaces*

$$H_\bullet(C_\bullet, \partial_\bullet) \cong H_\bullet(C^\Sigma_\bullet, \partial^\Sigma_\bullet). \tag{38}$$

As the proof of this theorem is quite complex we shall break it down it several steps. The first part of this is to utilize an idea by Whitehead where he reduces a CW cell pair while preserving its homotopy type.

DEFINITION 3.18. *Let* $(x^\star, y^\star) \in \Sigma$ *and define the **reduction*** $(\mathcal{P}^\star, \leqslant_\star)$ *with* $\mathcal{P}^\star = \mathcal{P} \setminus (x^\star, y^\star)$ *and* $w \prec_\star z$ *if either* $w \prec z$ *or* $w \prec y^\star \succ x^\star \prec z$. *This turns* $(\mathcal{P}^\star, \leqslant_\star)$ *into a graded partial order.*

Given a coparameterization F over $(\mathcal{P}, \leqslant)$ one obtains a new coparameterization $F_\star$ over $(\mathcal{P}^\star, \leqslant_\star)$ as follows:

- $F^\star(w) = F(w)$, and
- for each covering relation $w \prec_\star z$ we have the linear map $F^\star_{z,w}$ given by

$$F^\star_{z,w} := F_{z,w} - F_{y^\star, w} \circ F^{-1}_{y^\star, x^\star} \circ F_{z, x^\star}. \tag{39}$$

We then have that $F^\star$ coparameterization a new chain complex $(C_\bullet^\star, \partial_\bullet^\star)$ over the poset $(\mathcal{P}^\star, \leqslant_\star)$. Moreover Σ restricts to an acyclic partial matching $\Sigma^\star$ on $(\mathcal{P}^\star, \leqslant_\star)$.

PROPOSITION 3.19 ([CGN15]). *Given the restricted acyclic partial matching $\Sigma^\star$ defined above,*

(1) $\Sigma^\star$ is compatible with the reduced coparameterization $F^\star$, and
(2) the Morse data associated to $\Sigma^\star$ is identical to that of Σ.

PROOF ADAPTED FROM PROPOSITION 3.1 IN [CGN15]. For $x, y \in \Sigma^\star$ we have that $F_{y,x} = F^\star_{y,x}$. To see this, suppose to the contrary that it is not the case that $F_{y,x} = F^\star_{y,x}$. Then we must have that $F^\star_{y^\star,x}$ and $F^\star_{y,x^\star}$ do not vanish. However, this implies that $(x,y) \lhd (x^\star, y^\star) \lhd (x,y)$, a contradiction.

Now, to show the second point we first note that the critical elements of Σ and $\Sigma^\star$ are the same because $(x^\star, y^\star) \in \Sigma$. Thus, since $F(x) = F^\star(x)$ for $x \in M$, we have that $C_\bullet^\Sigma = C_\bullet^{\Sigma^\star}$. The final point is therefore to show that $\partial_\bullet^\Sigma = \partial_\bullet^{\Sigma^\star}$. If $\gamma^\star$ is any gradient path of $\Sigma^\star$, say

$$\gamma^\star = x_0 \prec_\star y_0 \succ_\star x_1 \prec_\star y_1 \succ_\star \cdots \succ_\star x_n \prec_\star y_n, \tag{40}$$

then it follows by the acyclity of Σ that there is at most one index $i \in \{0, 1, \ldots, n\}$ such that $(x_i, y_i) \lhd (x^\star, y^\star) \lhd (x_{i+1}, y_{i+1})$. Thus there are only two possibilities, either

(1) there is no index i such that the removed pair $(x^\star, y^\star)$ might fit in $\gamma^\star$, in which case $\gamma^\star$ is also a gradient path of Σ with $F^{\gamma^\star} = (F^\star)^{\gamma^\star}$, or
(2) there is a single such index i, in which case Σ may have both $\gamma^\star$ and the unique augmented path γ, which takes the form
$$\gamma = x_0 \prec y_0 \succ \cdots \prec y_i \succ x_i^\star \prec y_i^\star \succ x_{i+1} \prec \cdots \succ x_n \prec y_n, \tag{41}$$
as gradient paths.

In the second case it follows that $(F^\star)^{\gamma^\star} = F^{\gamma^\star} + F^\gamma$. Hence, in either case we have that the indexes agree which implies that $\partial_\bullet^\Sigma = \partial_\bullet^{\Sigma^\star}$. □

If we can show that homology remains unaffected when going from the coparameterization F to the coparameterization $F^\star$, then we can remove elements from Σ until $F^\star = F^\Sigma$ in which case we have our desired isomorphism $H_\bullet(C_\bullet, \partial_\bullet) \cong H_\bullet(C_\bullet^\Sigma, \partial_\bullet^\Sigma)$. To do this, we construct a chain map between $C_\bullet$ and $C_\bullet^\star$. For n we let ψ_n be given blockwise as follows: for $z \in \mathcal{P}_n$ and $w \in \mathcal{P}_n^\star$, let $\psi_n^{z,w} : F(z) \to F(w)$ be given by

$$\psi_n^{z,w} = \begin{cases} -F_{y^\star,w} \circ F^{-1}_{y^\star,x^\star}, & \text{if } z = x^\star \\ 1_{F(z)}, & \text{if } z = w \\ 0, & \text{else.} \end{cases} \tag{42}$$

Note that this definition implies that $\psi_n = 1_{C_n}$ for $n \neq \dim x^\star$.

LEMMA 3.20. *The morphism $\psi_\bullet : C_\bullet \to C_\bullet^\star$ is a chain map.*

DUAL OF PROOF OF LEMMA 3.2 [CGN15]. Let $z \in \mathcal{P}_n$ and $w \in \mathcal{P}^\star_{n-1}$. We shall show that the blocks of $\psi_{n-1} \circ \partial_n$ and $\partial_n^\star \circ \psi_n$ from $F(z)$ to $F^\star(w) = F(w)$ are equal. More specifically, we shall show that

$$\sum_{z' \in \mathcal{P}_{n-1}} \psi_{n-1}^{z',w} \circ F_{z,z'} = \sum_{w' \in \mathcal{P}_n^\star} F^\star_{w',w} \circ \psi_n^{z,w'}. \tag{43}$$

From Equation 42 we have that the left side of the equation above is nonzero only for $z' = w$ or $z' = x^\star$. The left sum therefore becomes $F_{z,w} + \psi_{n-1}^{x^\star,w} \circ F_{z,x^\star} = F^\star_{z,w}$.

If $z \neq x^\star$ then the right side of the sum is also equal to $F^\star_{z,w}$. Hence it suffices to show that the right side equals $F^\star_{z,w}$ when $z = x^\star$. In this case the right side becomes

$$\sum_{w' \in \mathcal{P}^\star_n} F^\star_{w',w} \circ \psi_n^{x^\star,w'} = -\sum_{w' \in \mathcal{P}^\star_n} F^\star_{w',w} \circ F_{y^\star,w'} \circ F^{-1}_{y^\star,x^\star}. \tag{44}$$

Expanding $F^\star_{w',w}$ and distributing over terms gives

$$-\sum_{w' \in \mathcal{P}^\star_n} F_{w',w} \circ F_{y^\star,w'} \circ F^{-1}_{y^\star,x^\star} + \sum_{w' \in \mathcal{P}^\star_n} F_{y^\star,w} \circ F^{-1}_{y^\star,x^\star} \circ F_{w',x^\star} \circ F_{y^\star,w'} \circ F^{-1}_{y^\star,x^\star}. \tag{45}$$

The second sum is zero since $x^\star \prec y^\star$ implies that there is no $w' \in \mathcal{P}$ such that $x^\star \prec w' \prec y^\star$. Then, as $\partial_\bullet$ is a boundary operator we have that $\sum_{w' \in \mathcal{P}} F_{w',w} \circ F_{y^\star,w'} = 0$ which implies that the first sum is equal to $F^\star_{z,w}$ as desired. $\square$

Dualizing the argument in a similar manner to above for the proof of Lemma 3.3 in [CGN15] we get a chain map $\phi : C^\star_\bullet \to C_\bullet$ which is defined blockwise for $w \in \mathcal{P}^\star_n$ and $z \in \mathcal{P}_n$ as the map $\phi_n^{w,z}$ with

$$\phi_n^{w,z} = \begin{cases} -F^{-1}_{y^\star,x^\star} \circ F_{w,x^\star}, & \text{if } z = y^\star, \\ 1_{F(z)}, & \text{if } z = w, \\ 0, & \text{else.} \end{cases} \tag{46}$$

Proof of Theorem 3.17. Combining the above results, the only thing that remains to show is that there is a chain homotopy between $\phi_\bullet \circ \psi_\bullet$ and the identity on $C_\bullet$ and similarly with $\psi_\bullet \circ \phi_\bullet$ and the identity on $C^\star_\bullet$. Now, $\psi_\bullet \circ \phi_\bullet = 1_{C^\star_\bullet}$ and hence it only remains to find a chain homotopy $\Theta : C_\bullet \to C_{\bullet+1}$ between $\phi_\bullet \circ \psi_\bullet$ and $1_{C_\bullet}$. To this end, let $\Theta_n : C_n \to C_{n-1}$ be given blockwise as the map

$$\Theta_n^{z,z'} = \begin{cases} F^{-1}_{y^\star,x^\star}, & \text{if } z' = y^\star \text{ and } z = x^\star \\ 0, & \text{else.} \end{cases} \tag{47}$$

We must verify that $\Theta_{n-1} \circ \partial_n + \partial_{n+1} \circ \Theta_n = 1_{C_n} - \phi_n \circ \psi_n$. Using the definition of ψ_n, ϕ_n, and Θ_n blockwise then finishes the proof. $\square$

Lemma 3.21. *For $\theta : F \to G$ a map of Σ-compatible coparameterizations there is an induced map $\theta^* : F_* \to G_*$ and induced chain map $\Theta^* : C^*_\bullet \to C'^*_\bullet$ such that*

(1) for $w \in \mathcal{P}^*$ we have $\theta^*_w = \theta_w$,
(2) the maps $\psi^F : C_\bullet \to C^*_\bullet$ and $\psi^G : C'_\bullet \to C'^*_\bullet$ are compatible with Θ^* in the sense that the following diagram commutes

$$\begin{tikzcd} \bigoplus_{z \in \mathcal{P}_n} F(z) \arrow[r, "\Theta_n"] \arrow[d, "\psi^F_n"'] & \bigoplus_{z \in \mathcal{P}_n} G(z) \arrow[d, "\psi^G_n"] \\ \bigoplus_{w \in \mathcal{P}^*_n} F(w) \arrow[r, "\Theta^*_n"'] & \bigoplus_{w \in \mathcal{P}^*_n} G(w), \end{tikzcd}$$

(3) there is an isomorphism $\mathrm{im}(H_\bullet \Theta^*) \cong \mathrm{im}(H_\bullet \Theta)$.

Proof. Let θ^* be as specified. If $s \in F(z)$ and $z \neq x^*$ then commutativity follows directly as θ is a natural transformation. Now, if $z = x^*$ then we have that the $G(w)$ part of the result is given by

$$\begin{aligned}(\psi^G)_n^{x^*,w}(\theta_{x^*}(s)) &= -G_{y^*,w}(G^{-1}_{y^*,x^*}(\theta_{x^*}(s)))\\ &= -G_{y^*,w}(\theta_{y^*}(F^{-1}_{y^*,x^*}(s)))\\ &= \theta_w(-F_{y^*,w}(F^{-1}_{y^*,x^*}(s)))\\ &= \theta_w((\psi^F)_n^{x^*,w}(s))\end{aligned}$$

showing that ψ^F and ψ^G are compatible with Θ^*. Next, to see that Θ^* is a chain map we need to show that $\Theta^*_{n-1} \circ \partial^*_n = \partial'^*_n \circ \Theta^*_n$. To do this we precompose with ψ^F_n and get that

$$\begin{aligned}\Theta^*_{n-1} \circ \partial^*_n \circ \psi^F_n &= \Theta^*_{n-1} \circ \psi^F_{n-1} \circ \partial_n\\ &= \psi^G_{n-1} \circ \Theta_{n-1} \circ \partial_n\\ &= \psi^G_{n-1} \circ \partial'_n \circ \Theta_n\\ &= \partial'^*_n \circ \psi^G_n \circ \Theta_n\\ &= \partial'^*_n \circ \Theta^*_n \circ \psi^F_n.\end{aligned}$$

Then, since ψ is an epimorphism we get the desired equality $\Theta^*_{n-1} \circ \partial^*_n = \partial'^*_n \circ \Theta^*_n$.

Finally to see that there is an isomorphism $\mathrm{im}(H_\bullet\Theta^*) \cong \mathrm{im}(H_\bullet\Theta)$ remember that ψ and ϕ are inverses of each other when taking homology, i.e., $H_\bullet\phi^F = (H_\bullet\psi^F)^{-1}$. Now, as taking homology is functorial, we have that $H_\bullet\Theta^* \circ H_\bullet\psi^F = H_\bullet\psi^G \circ H_\bullet\Theta$. Putting it together we have

$$H_\bullet\Theta^* = H_\bullet\psi^G \circ H_\bullet\Theta \circ H_\bullet\phi^F, \tag{48}$$

and thus $H_\bullet\psi^G$ provides an isomorphism between the two images. □

Proposition 3.22. *Let $(F_\bullet, \theta_\bullet)$ be a filtration of coparameterization and $0 \leqslant k \leqslant l$. We then have that*

$$\mathrm{im}(H_\bullet(\Theta^k)^\Sigma \circ \cdots H_\bullet(\Theta^l)^\Sigma) \cong \mathrm{im}(H_\bullet\Theta^k \circ \cdots H_\bullet\Theta^l). \tag{49}$$

Proof. Fix $0 \leqslant k$. We then do induction on $l \geqslant k$. The base case $l = k$ was essentially[5] covered in the previous lemma and hence we focus on the inductive step. Thus, suppose it holds for $l \geqslant k \geqslant 0$. We want to show it for $l+1$. Now, using the same argumentation as in the lemma, the induction hypothesis implies that

$$H_\bullet(\Theta^k)^\Sigma \circ \cdots H_\bullet(\Theta^l)^\Sigma \cong H_\bullet\psi^{k+1} \circ H_\bullet\Theta^k \circ \cdots H_\bullet\Theta^l \circ H_\bullet\phi^l. \tag{50}$$

Then, as $H_\bullet(\Theta^{l+1})^\Sigma = H_\bullet\psi^l \circ H_\bullet\Theta^{l+1} \circ H_\bullet\phi^{l+1}$ and $H_\bullet\phi^l \circ H_\bullet\psi^l = 1$, we get that

$$H_\bullet(\Theta^k)^\Sigma \circ \cdots H_\bullet(\Theta^{l+1})^\Sigma \cong H_\bullet\psi^{k+1} \circ H_\bullet\Theta^k \circ \cdots H_\bullet\Theta^l \circ H_\bullet\phi^{l+1}. \tag{51}$$

Hence $H_\bullet\psi^{k+1}$ is an isomorphism between the two images. □

3.4. The simplicial complex case. We finish off this section by utilizing the results from the previous subsection to construct a filtered Morse chain complex and show that it has the same homology as the one presented in Definition 3.3.

Let $(\mathcal{F}_\bullet, \phi_\bullet)$ be a filtration of $\mathbf{Vec}_k$ valued cosheaves on K which is compatible with some acyclic partial matching Σ on K.

[5]One just has to iterate the proof from Θ^* to Θ^Σ

Definition 3.23 (Filtered Morse chain complex). *The **filtered Morse chain complex** associated to* $(\mathcal{F}_\bullet, \phi_\bullet)$ *is defined to be the chain complexes* $(\check{C}^\Sigma_\bullet(\mathbf{St}_K; \mathcal{F}_n), \partial^\Sigma_\bullet)$ *and the maps*

$$\Psi^n_\bullet : \check{C}^\Sigma_\bullet(\mathbf{St}_K; \mathcal{F}_n) \to \check{C}^\Sigma_\bullet(\mathbf{St}_K; \mathcal{F}_{n-1}) \tag{52}$$

where $\Psi^n_j := \bigoplus_{\sigma \in M_j} \phi_{n,\mathbf{st}(\sigma)}$.

Lemma 3.24. *The maps* $\Psi^n_\bullet$ *are chain maps for all* $n \in \mathbb{N}$ *and hence we have induced maps on homology*

$$\cdots \xrightarrow{\check{H}^\Sigma_\bullet \Psi^{n+1}_\bullet} \check{H}^\Sigma_\bullet(\mathbf{St}_K; \mathcal{F}_n) \xrightarrow{\check{H}^\Sigma_\bullet \Psi^n_\bullet} \check{H}^\Sigma_\bullet(\mathbf{St}_K; \mathcal{F}_{n-1}) \xrightarrow{\check{H}^\Sigma_\bullet \Psi^{n-1}_\bullet} \cdots \xrightarrow{\check{H}^\Sigma_\bullet \Psi^1_\bullet} \check{H}^\Sigma_\bullet(\mathbf{St}_K; \mathcal{F}_0). \tag{53}$$

Proof. In each step of removing elements when going from $C_\bullet$ to $C^\Sigma_\bullet$ the maps Θ^* are all chain maps and so it follows that Θ^Σ is too. Hence, as $\psi^n_\bullet = \Theta^\Sigma$ in this scenario, the result follows. □

Proposition 3.25. *The persistent homology of the filtered Morse chain complex for* K*, as described previously, is the same as the persistent homology for the Čech open cover* $\mathbf{St}_K$.

Proof. Let $(F_\bullet, \theta_\bullet)$ be the associated filtration of coparameterizations associated to $(\mathcal{F}_\bullet, \theta_\bullet)$. Now, by definition of being a filtration of coparameterizations we know that we can use this filtration as a replacement for the filtration of cosheaves and get the exact same persistent homology. The claim then follows directly from Proposition 3.22. □

Now, the `Scythe` algorithm in [CGN15] can be suitably dualized to provide an algorithm for finding a Morse coparameterization of a cosheaf on a finite poset. Combining this with Proposition 3.25 we have a way to speed up computing the persistent homology of a sequence of cosheaves over a finite simplicial complexes. The key assumption being made here is that simplicial complex is finite as this means each coparameterization can be represented in memory. In fact, our results in this essay implies something a bit more general; using a dualized version of `Scythe` there is an efficient way of computing persistent homology of a sequence of coparameterizations over a finite poset.

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